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Beyond the Representation Given – The Parabola and Historical Metamorphoses of Meanings

Christer Bergsten

Introduction

As well known, René Thom suggested that mathematics education should be founded on meaning rather than rigour. As if one could not have both. Having neither would not be an attractive option. Of course, it all depends (doesn't it always?) on what meaning (sic!) is given to these terms (i.e. meaning and rigour), and what goals of mathematics education are being adhered to. One also needs to interpret the claim made by Thom through its context – a presentation at ICME2 in 1972 about 'modern mathematics' during the time of the 'New Math' and its emerging problems:

> *The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects*. (Thom, 1973, p. 202, italics in original)

The formalism of the 'modern mathematics' is not *natural* in education, Thom argues, necessary only in numerical and algebraic computations, and the importance of rigour in mathematics is over-estimated. Meaning is linked to consciousness and existence of the mathematical objects in the mental world. Contrasting algebraic language with Euclidean geometry, Thom finds that the move to

> eliminate elementary geometry to make room for calculus and linear algebra, has little to recommend it psychologically, because the algebraic objects (the symbols) are too poor semantically to permit themselves to be understood directly as is the case with a spatial figure. (ibid., pp. 207-208)

By its power to express structure,

> the spirit of geometry circulates almost everywhere in the immense body of mathematics, and it is a major pedagogical error to seek to eliminate it (ibid., p. 208).

In contrast to a kind of secondary meaning *given* to algebraic symbols, geometrical objects appeal *directly* to intuition, one could argue (possibly in line with Thom), by their figurative appearance to the senses, providing a diagrammatic structure on which meanings can be built. By deleting elementary (classical?) geometry from the school curriculum, Thom argues, the "spirit of geometry" that enters mathematical reasoning also when the focus is not geometry itself, would jeopardize the presence of meaning generally in mathematics.

Now, the problem of meaning did not disappear with the New Math. 'Formal' mathematics in the sense of numerical and algebraic computations fill up much of students' work in elementary mathematics classrooms today, for many coupled with a double meaninglessness: in the symbols used (as Thom noted) but also the rules by which they are operated on. This chapter, though, asks the question what happens to this "spirit of geometry" with its *algebraization*, turning it into a study of coordinates and graphs of functions, in terms of the 'meaning' it provides. Considering for example the parabola, it is not only problematic, according to Bartolini Bussi (2005) to define a conic section algebraically, "*it is not possible to build the meaning of conics through only a one-sided approach*, as, for instance, through the most widespread algebraic definition" (p. 39). In line with this quote, it would also be impossible to remain within the Euclidean geometry system of reasoning tools when aiming for *the* meaning of conics. If there is such a thing. Always lurking behind any discussion of meaning in mathematics, and its education, is Russell's (1901) comment that mathematics "may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true". Such statement stems from a specific philosophy of mathematics, an unavoidable link also in education as Thom (1973) reminds us about: "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." (p. 204)

As a contribution to the discussion of meaning in mathematics education, this chapter will draw on the example of the parabola to investigate, through an historical lens, the relation between representations and meanings of a mathematical object. After a short introductory theoretical account on this relation and some mathematical preliminaries, a commented search for the life of the parabola through the history of mathematics is presented. This is followed up by a discussion of the elusive notion of meaning. In the final part of the chapter some issues related to meaning in the teaching of the parabola in school mathematics will be raised, drawing on the notion of praxeology (or mathematical organisation).

The parabola and the other conics have been studied extensively in the history of mathematics as well as in mathematics education. Is there really more to say? Indeed, this chapter also draws on an earlier presentation (with the same title as this chapter[1]) in a discussion group at PME in Bergen in 2004 (for a written version of this presentation, see Bergsten, 2004), and some parallels in Bartolini Bussi (2005). However, the exercise in this chapter aims to elaborate in some detail on different representations of a specific mathematical object (the parabola) in a search for some refractions of the relation between meaning and rigour in school mathematics.

Representations and meaning

Thom's claim about the necessity of focusing on 'meaning' rather than 'rigour' in mathematics education, with reference to a mental existence of mathematical objects, invites some initial reflections on 'meaning'. Indeed, conceptions of meaning rest on "presuppositions concerning the relationship between the cognizing subject and the object of knowledge" (Radford, 2006, p. 40). The ontological issue whether this *object of knowledge* refers to something that has an 'existence' independently of the knower, providing an essence of objectivity of the knowledge necessary to earn it a status of knowledge that can be shared with others and not exclusively a purely subjective construction, has been discussed also within mathematics education. Radford (2006), for example, makes the distinction between an *objective referential*

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[1] The title alludes to Bruner (1973) who argues that "when one goes beyond the information given, one does so by virtue of being able to place the present given in a more generic coding system and that one essentially reads off from the coding system additional information either on the basis of learned contingent probabilities or learned principles of relating material" (p. 224).

approach, “which locates mathematical objects outside semiotic activity” and “played an important role in the shaping of the pedagogical scene at the turn of the 20th century” (p. 41), a *discursive approach* in which mathematical objects are constituted as discursive objects within the mathematical discourse (see e.g. Dörfler, 2002; Sfard, 2008), and *a semiotic-cultural approach* where “ideas and mathematical objects […] are conceptual forms of historically, socially, and culturally embodied reflective, mediated activity” (Radford, 2006, p. 42) with an emphasis on the role of *semiosis*. The latter approach describes *objectivity* as contextual and cultural, in which “the relationship between the observer and that which is observed is a culturally mediated one” (ibid., p. 60), where “the whole arsenal of signs and objects that we use to make our intentions apparent” constitute the *semiotic objectification of knowledge* (see also Radford, 2003). This is put in contrast to Peirce’s view where ultimately the ‘true’ objectivity resides at the horizon beyond an endless process of semiotic chaining, steered by his *pragmatic maxim*, and where “meaning is the relation that links one sign to the next sign (i.e. its interpretant) in a semiotic chain” without taking into account social praxis and human interaction (Radford, 2006, pp. 44-45). For Radford then, meaning is both a subjective and a cultural construct, linked to the individual’s “personal history and experience” but also “prior to the subjective experience, the intended object of the individual’s intention (l’object visé) has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it” (ibid., p. 53).

Representations are also in the centre for Damerow (2007) in his account for the connection between historical developments and individuals’ cognition, when outlining a theory of the historical development of arithmetical thought. “[B]asic structures of logico-mathematical thought”, he writes,

> are developed by the individual growing up in confrontation with culture-specific challenges and constraints under which the system of action have to be internalised. The challenges are embodied in the material means of goal-oriented or symbolic actions that are shared representations of logico-mathematical structures of mental models. (p. 22)

He makes a distinction between first- and second- (or higher) order external representations of mental models. Here the *first-order representations* are

> material representations of real objects by symbols or by models composed of symbols and rules of transformation, with which essentially the same actions can be performed as with the real objects themselves (p. 25)[2]

Among the examples provided, constructions with compass and ruler are first-order representations of geometrical figures (in the Euclidean plane), useful for locating objects, their relations and movements. Regarding the function of first-order representations, actions on the representations can be performed more easily and freely than actions on the objects themselves, to foresee the results of the real actions. These symbolic actions also "initiate the construction of the same mental models as actions with the real objects they represent" (p. 26). While the same symbols are used in different contexts, their meanings become differentiated. To *represent mental models*, however, *second- (or higher-) order representations* are needed, consisting of

> symbols or models composed of symbols and rules of transformation which correspond to the operations of the abstract mental model that controls the actions performed with the real objects (p. 26)

To be used properly, a second-order representation needs to be related to the real objects and actions that are to be represented by its symbolic system by "assimilating them to the mental model that gives the external representation its meaning" (p. 26). For example, some theorems in Euclidean geometry (e.g. the Pythagorean theorem) are second-order representations as they do not relate to concrete geometric figures but to discursively defined mathematical objects within a deductive framework. Through second-order representations meta-cognitive mental models "may be constructed as higher-order representations by reflective abstraction" (p. 27). As the external (material) representations are historically transmitted, they can be used for the reconstruction of the mental models (reflected meanings) they represent.

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[2] This description parallels the notion of a *genetic mathematical form* (see Bergsten, 1999), referring to a symbolic expression that has a form (structure) isomorphic to a material realisation of the signified.

In school mathematics, students constantly encounter and are expected to construct meanings through work with exercises on historically transmitted representations of mathematical objects. Here the use of higher-order representations is ubiquitous but often without established links to lower-order representations and their "cultural values and theoretical content", leaving potential gaps in students' meaning making processes.

Why look at the parabola?

Consider the following claim, having in mind its relevance for meanings students may experience in their mathematics studies: *In today's school mathematics, concepts and methods are often treated in isolation, with only surface level theoretical embedding within a mathematical sub-domain.* This very general statement, endowed, though, with a considerable degree of specification, is here not supported by any reference or argument but nevertheless taken as the starting point for the chapter.

The treatment of concepts and methods is, at least in the Swedish secondary school context that will be referred to here, more or less dominated by algebraic tools. However, as often pointed out, many students do not 'achieve well' in algebra, in neither of the aspects mentioned above. As a consequence, an image of mathematics as disconnected and difficult to understand may emerge in students. Applications used in teaching to enhance motivation then easily get the character of decorations rather than parts of an integrated knowledge structure. One example that well illustrates this didactic phenomenon is the parabola. Problems and techniques related to this 'classical' curve are ubiquitous in school mathematics:

- the concept of square root, often used as a basis for the extension of the number concept from rational to real numbers;
- the solution of quadratic equations, often used as a basis for the extension of the number concept from real to complex numbers;
- a prototype for polynomials and the connection between zeros and factoring of polynomials;
- second degree polynomials often serve as a prototype for the study of derivatives and optimisation problems;

- a prototypical example for mathematical modelling, e.g. of reflections (parabolic antennas) and projectile motion;
- etc.

Evidently, a student's conceptualisation of a parabola is framed by how it is described, defined, treated, applied, etc., in its educational context. The term *concept image* has been coined to capture the (non-static) mental outcome of this process (Tall & Vinner, 1981), indicating the "existence of the mathematical objects in the mental world" (quote from this text, in the introduction above). A key role is here taken by the *representations* used, and how the *mathematical object* in focus is being objectified by their mediation (Radford, 2003). In today's school mathematics, different representations of a mathematical object (such as the parabola) all live in a mixed world of mathematical ideas and tools, not always treated by a systematic approach, as seen from the students' perspective. Here the asymmetry (in specialised knowledge) between the teacher and the student comes into play, as the teacher may have links between mathematical ideas (implicitly) in mind while presenting a specific mathematical object that are not available to the students. The parabola, with its long cultural history, is such an object.

Preliminaries

In order to found an initial meaning of 'parabola', I will provide a short reminder of some preliminaries based on Figure 1, employing two elementary mathematical tools: the Pythagorean Theorem and mean proportion. I will focus on how some basic properties of the two-dimensional curve named parabola can be derived from its original three-dimensional construction from cutting a right-angled cone by a plane parallel to its surface,

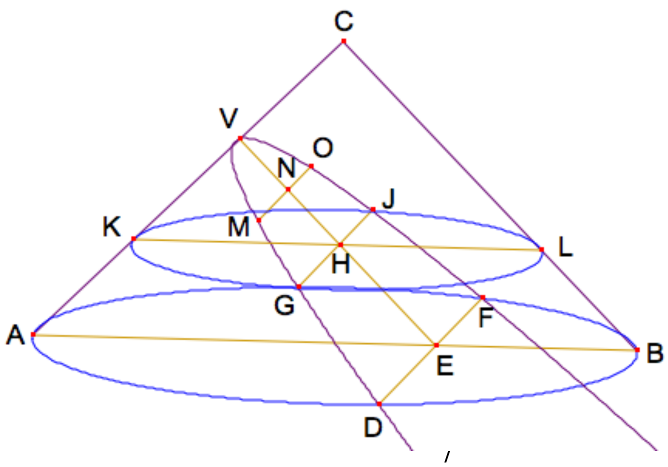

Figure 1. A parabolic cut from a right-angled cone

In Figure 1 AB is the diameter of the circle produced by a plane cutting a circular right-angled cone (with vertex C and right angle ACB) perpendicular to its axis. The section DVF (the *parabola*) has been produced by a plane cutting the cone parallel to BC and perpendicular to AB. If the distance CV is of length a, then HL and EB are both of length $a\sqrt{2}$. The point H should here be imagined as moving along VE. In the circle with diameter KL, GH is the mean proportion of KH and HL. Thus, with GH having length x and VH length y, KH has length $y\sqrt{2}$, and it follows that $x^2 = y\sqrt{2} \cdot a\sqrt{2}$, i.e. $x^2 = 2ay$. In this way, the quadratic function is intrinsically linked to the parabola.[3]

Now consider the specific value of x for which x and y are equal (GH = VH), i.e. $x^2 = 2ax$. The non-zero solution $x = 2a$ is in Figure 1 drawn as the segment DE, and MN has half this length ($x = a$). Again, by mean proportion in the circle at 'level' N (not drawn in Figure 1), one finds that $\mathrm{VN} = \frac{a}{2}$, which is a fourth of DE, and also of VE. The point N is of special interest considering the distance d from the arbitrary point G, on the parabola, to N (see also Figure 2):

$$d^2 = \mathrm{GH}^2 + \mathrm{HN}^2 = x^2 + \left(\frac{x^2}{2a} - \frac{a}{2}\right)^2 = \frac{1}{4a^2}\left(x^2 + a^2\right)^2$$

It follows that $\mathrm{GN} = \mathrm{VH} + \frac{a}{2}$. This is the well-known *equidistance property* of the parabola, illustrated in Figure 2 where the plane containing the parabolic section in Figure 1 is laid out.

[3] In terms of areas, the formula states that the square on GH equals the rectangle with sides VG and twice CV. As CV is a constant, a relation is thus found between VH and GH for an arbitrary point G on the parabola.

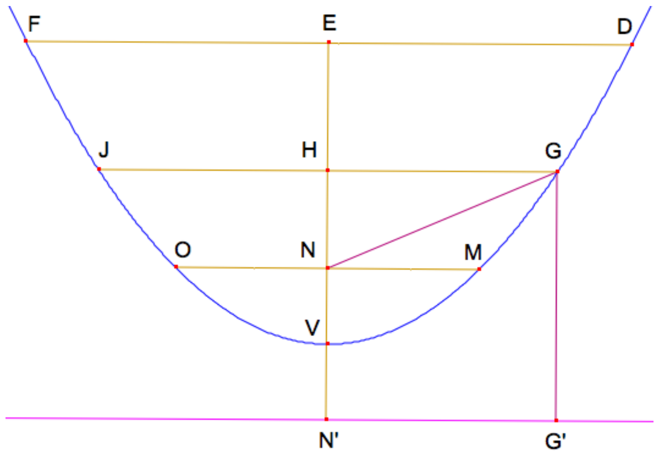

Figure 2. The equidistance property of a parabola

In Figure 2, VN' is constructed equal to VN and thus, as shown above, GN = HN' = GG'. The point D was chosen so that ED = EV and M chosen so that OM = ED. The chord OM, passing the *focus* N, is often referred to as *latus rectum* ('straight line') and its length (equal to 2*a* in Figure 1) as the *parameter* of the parabola. As seen above, the length of latus rectum is four times the distance from the focus to the vertex. The line through N' and G' is known as the *directrix*.

Historical origins

Tracking the history of a mathematical object reveals a double influence on its development – from inside and outside the pure mathematical realm. In this part of the chapter, I will provide a short account of the historical progression of the mathematical object *parabola*, within classical geometry, analytic geometry, and dynamic geometry software. These 'steps' parallel those described by Bartolini Bussi (2005, pp. 40-42) for the historical development of the conics in mathematics and society. From a semiotic point of view, a chain of metamorphoses of meanings will be described. My interest is on how this becomes critical as the mathematical object by the social process of didactic transposition is turned into an object for teaching and learning.

Archimedes

In the works of Euclid, Archimedes and Apollonius the parabola is a geometric object, defined rhetorically and analysed by the tools of constructive and deductive (Euclidean) geometry. Historically, the conic sections first appeared, it is believed, in the attempts of Menaechmus (approx. 350 B.C.) to solve the problem of duplicating the cube (Eves, 1983, p. 80). Later, in his famous treatise on the quadrature of the parabola, Archimedes speaks of the parabola as ”a section of a right angled cone” (Archimedes, 1952, p. 527)[4] and refers to ”the elementary propositions in conics which are of service in the proof”, three of which are referenced to work by Euclid and Aristaeus. One of these ”elementary propositions” listed is of special interest in a discussion of meanings related to the parabola (see Figure 3)[5]:

> If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q′ on the parabola straight lines be drawn parallel to the tangent at P meeting PV in V, V′ respectively, then $PV : PV' = QV^2 : Q'V'^2$.

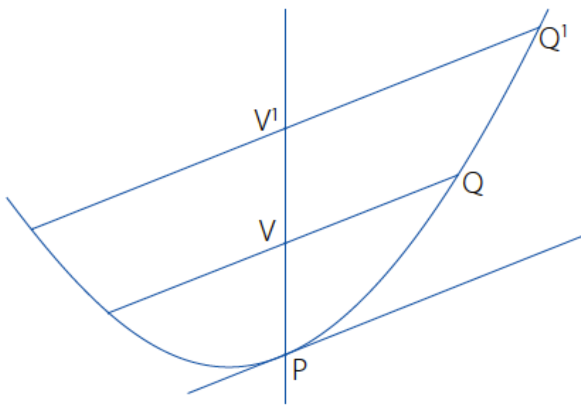

Figure 3. An illustration to a proposition referred to by Archimedes (1952, p. 528)

[4] By this definition a parabola is the section produced by a plane cutting a right angled cone at a right angle to an element of the cone (cf. the preliminaries above).

[5] The power notation on QV is used by the editor of Archimedes’ text to refer to the square with each of its sides equal to QV. Figure 3 does not appear in Archimedes’ text but a similar diagram with only one chord related to the following proposition: “If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if QQ’ be a chord parallel to the tangent to the parabola at P and meeting PV in V, then QV = QV’.” (Archimedes, 1952, p. 528)

This proposition is thus by Archimedes taken as "proved in the elements of conics" (ibid.). However, it appears later in Apollonius' *Conics* along with a proof (see below).

Here the reader might wonder about the purpose of this chapter, obviously placed in the educational domain but now elaborating on a (rather standard) historical *Rückblick*, perhaps asking as did Fried (2013, p. 2), "is there really a distinct educational relationship to our mathematical past?". A rationale for a mathematical elaboration of this historical part of the text for mathematics educators, apart from the intended discussion of meaning, is that this piece of mathematics may have some unexpected relevance when recontextualised from the point of view of education. For example, the proposition quoted might be an eye-opener, questioning the privilege given the orthogonal Cartesian coordinate system in school mathematics and its 'formatting' over the imagined range of mathematical relationships. In the paragraph that will follow, however, the relevance is presented as inherent in the proposition itself by way of bridging different mathematical domains, thus providing a potential counteract against the fragmentation of school mathematical knowledge highlighted in the introduction. As a clarification of this claim, by allowing a semiotic jump when interpreting the proportional equality $PV : PV' = QV^2 : Q'V'^2$ as an algebraic equation of quotients, the introduction of the variables $y = PV$, $y' = PV'$, $x = QV$, $x' = QV'$ in the expression brings it into the form $\frac{y}{y'} = \frac{x^2}{x'^2}$. This implies that $\frac{y}{x^2} = \frac{y'}{x'^2} = k$ and thus $y = k \cdot x^2$. It should be noted here that the lines PV and QV are not necessarily perpendicular.

Application of areas – a reconstruction

The 'well known' proposition referred to by Archimedes relates the parabola, defined as a conic section, in modern mathematical terminology and within another mathematical domain, to the quadratic function. The initial constructions involved in relating the parabola to a square, in line with the proposition above, may have been built on the Pythagorean method of applications of rectilinear areas[6]. As an illustration of this method within the present context, the construction (or reconstruction?) presented in Figure 4 is offered, which draws on Euclid I.43[7].

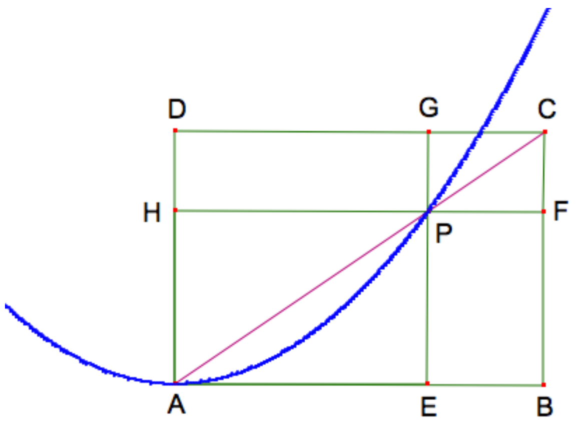

Figure 4. A construction of a parabola by the application of areas of rectangles (from Bergsten, 1998, p. 170).

In Figure 4, AB is a given segment (parameter). Drawing on Euclid I.43, an application of the square AEGD (standing on the variable segment AE) to a rectangle standing on AB will produce the point P (as the intersection point of EG and the diagonal AC) and the rectangle ABFH (with equal area as the square). As the point E is moving on AB or its extension the point P will move along a parabola. Using algebraic notation, with AE = x, EP = y, and AB = p, we have $py = x^2$.

The previous paragraph seems to be an attempt to "revoice" the mathematics of the past (cf. Boero, Pedemonte, & Robotti, 1997) in a way that makes the reader 'recognise' the common parabolic icon vertically placed on orthogonal coordinate axes. However, as the method of application of areas works equally well in the non-orthogonal case, Figure 5a below better illustrates this interpretation (i.e. meaning making) of the proposition mentioned by Archimedes. When the rhombus on AE is applied on the parallelogram on AB (here the length of AB is a fixed parameter), the parabolic curve appears as the locus of the point P as E is moved along AB or its

[6] See e.g. Sardelis and Valahas (2012). The method of application of areas is described by Proclus (ibid., p. 1). The names that Apollonius introduced for the conic sections (parabola, ellipse, hyperbola) have their origin in such applications (see e.g. Eves, 1983, p. 128).

[7] An online version of the *Elements* is available at http://farside.ph.utexas.edu/euclid/Elements.pdf

extension. This construction of the parabola thus draws on the property of a parabola stated in the proposition mentioned by Archimedes above[8]. Figures 4 and 5a were constructed by using geometry dynamic software employing the drag mode and the trace function to display, in modern mathematical terms, the functional relationship between the independent variable (AE) and the dependent variable (EP, equal to AH). The parameter AB shapes the parabola as more or less 'open'.

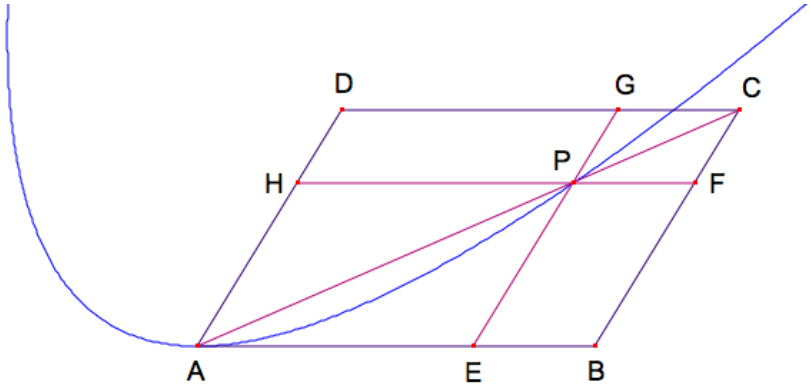

Figure 5a. A construction of a parabola by the application of areas of parallelograms[9]

For the 'left part' of the parabola in Figure 5a, Figure 5b illustrates the symmetric construction (where AB' equals AB, and AE' equals AE). By the drag mode in a dynamic geometry software this 'leaning' parabola can be straightened up to the

[8] The method presented here for the construction of (points on) a parabola differs from the "conceptual reconstruction" in Sardelis and Valahas (2012), which builds on the (standard) construction of a mean proportion in Euclid II.14. In terms of the mathematical language of functions, Sardelis and Valahas construct the *square root function* while Figure 4 (and Figure 5a) constructs the (inverse) *quadratic function* (though both by the application of areas). The "reconstruction" in Sardelis and Valahas (2012) is more aligned to the formulation in Apollonius (in connection to Figure 6) even though they argue that their "conceptual reconstruction supports the view that the conics were most probably discovered as plane curves by fusing the method of the application of areas with the concept of locus, long before Apollonius studied them as conic sections" (https://scirate.com/arxiv/1210.6842). Their method also covers an application of areas for the other two conics (drawing on constructions in Euclid). It should be noted that the method used in Figure 5a, based on the proposition presented in Archimedes' text (cf. above) supports the same view, and can also easily be extended to the other two conics with the use of Euclid II.14 for one step of the construction. However, the margin is too small to present these constructions here.

[9] This figure may be called a *parabelogram*. In the formula $y = k \cdot x^2$ for the parabola in Figure 5a, where AE = x and AH = y, one has $k \approx 0.2$. The size of the constant k does not depend on the leaning angle of the parallelogram but only on the length of the parameter AB (an increase of AB produces a decrease of k).

rectangular case (such as in Figure 4) by dragging the line defining the ‘upward’ direction of the parallelograms (cf. Figure 8 and the construction of the focus).

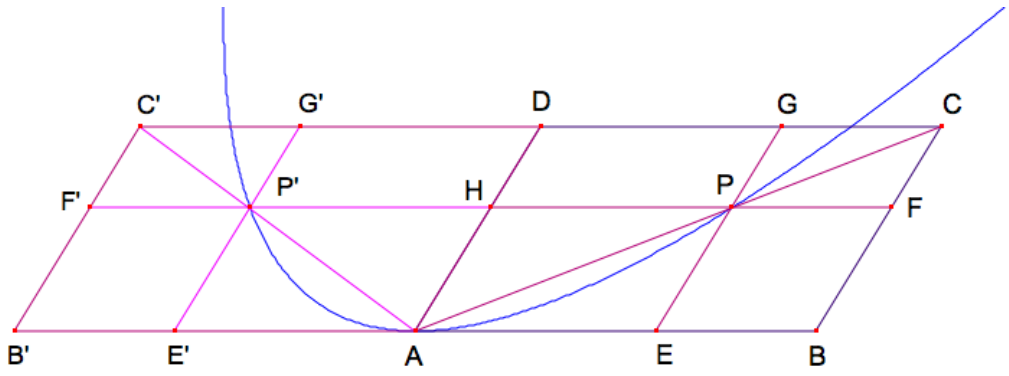

Figure 5a. Symmetry in the construction of the parabola by the application of areas

The principle upon which the construction draws can make the individual experience a meaning of the resulting mathematical object through the performance of the construction.

Apollonius

Unfortunately, in Archimedes’ text one does not find how the conception of a parabola as a specific conic section produces the property (proposition) quoted above and illustrated in Figure 1, as the work referenced is lost. This, however, one finds in Apollonius’ *Conics*. Apollonius defines a ‘diameter’ of a section as a line passing through the midpoints of parallel chords, and the ‘axis’ of the section as the diameter which is perpendicular to parallel chords. His definition of ‘parabola’, contained in a proposition with some similarities to the one given above, though, “is long because more than a statement of a claim, it is the description of a diagram” (Fried, 2007, p. 216). This certainly is true also for his definition of ‘cone’ (see Apollonius, 1952, p. 604). Apollonius’ definition of a parabola is as follows (see Figure 6):

> If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle, then if any straight line which is drawn from the section of the cone to its diameter

parallel to the common section of the cutting plane and of the cone's base, will equal in square the rectangle contained by the straight line cut off by it on the diameter beginning from the section's vertex and by another straight line which has the ratio to the straight line between the angle of the cone and the vertex of the section that the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola. (From Apollonius' *Conics I*, in Apollonius, 1952, p. 615)

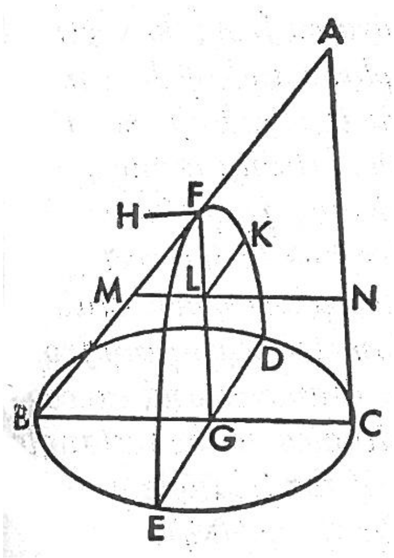

Figure 6. Apollonius' definition of a parabola (from Apollonius, 1952, p. 615)

With reference to Figure 6 (where the diameter FG is assumed to be parallel to AC). Apollonius defines the parameter FH (perpendicular to FL) by relating it to the shape of the cone through the requirement $FH : FA = BC^2 : BA \cdot AC$.[10] He then proves, by the use of elementary Euclidean propositions on triangle similarity, the proposition $LK^2 = LF \cdot FH$, corresponding to the application of areas shown in Figure 4 (where the parameter AB has replaced the parameter FH in Figure 6, and chosen independently as there is no cone involved; different lengths of AB would correspond to different vertex angles of the cone). From this proposition (Proposition 11 in Conics I), the proposition referred to by Archimedes and illustrated in Figure 3b immediately follows (as Proposition 20 in Conics I).

[10] In Figure 6, FG is parallel to AC, MN is parallel to BC, and DE is perpendicular to BC.

To Apollonius' proof as presented in Calinger (1995), the editor makes the note that "It was Apollonius's most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the *application of areas*" (ibid., p. 165; italics in original). This was however not a defining property as it follows a "then". The focal property is not discussed for the parabola but may be inferred as limit cases from such discussions about the hyperbola and the ellipse (see e.g. the note in Apollonius, 1952, p. 788). The terminology used, based on drawings where the size of segments are supposed to vary, have by some been seen as predecessors of both the coordinate system of analytic geometry and the concepts of variable and parameter as used in the study of functions (see e.g. Charbonneau, 1997; cf. Figure 4), as is common with a gaze on history from modern mathematics. As an historian refraining form employing such gaze, Fried (2007) notes that "the parabola, for Apollonius, is essentially a geometrical object and it is tied to the cone" (p. 216). However, as seen above (see also the quote above by Calinger, 1995, p. 165), by the observation of the application of areas involving a square, the parabola could 'leave' the cone and continue its life independently in a two-dimensional world, even more so after the 'discovery' of the equidistance property[11]. This way, with Apollonius the 'meaning' of a parabola as a purely geometric object can be anchored in the cone, by its construction as a section, or in the application of areas by the identification of a square and a rectangle within the plane of the parabolic curve. A new basis for a potential *change of meaning* appeared again later with Pappus: the 'meaning' of a parabola could now be anchored on the equidistance property, still though within the same representational register of figurative configurations that can be drawn/visualized as a basis for reflection and reasoning. Borrowing the terms *treatment* and *conversion* from Duval (2006), a *treatment* within the same semiotic register may incur adjustment or change of meaning, while a *conversion* to another register will imply a

[11] That the parabola (as defined by Apollonius) has the equidistance property (to a *focus* point and a *directrix* line) was shown by Pappus (Kline, 1972).

metamorphosis[12] of meaning. The latter could also be seen as paradigmatic shift, as will be the topic for the next section (sic!).

To arrive at the first *historical metamorphosis* a time travel of some 1850 years is now made from one line to the next (a jump also made by Bartolini Bussi, 2005, in her historical account of "the meaning of conics"), creating a gap in the coherence of the textual exposition as it leaves out the long term developmental process of the semiotics of the mathematical signs involved. This, however, does not affect the arguments made. It is also claimed that after Pappus "no significant progress in the study of conic sections had been made until the work of Kepler" (Koudela, 2005, p. 198).

A paradigmatic metamorphosis

After the advent of analytic geometry, most notably in the (independent) work of Descartes and Fermat, where the more modern algebraic notation in Descartes' book *La géomètrie* from 1637 was the most influential, a line (curve) could be described by the relation between coordinates (referring to distances along given directions) and studied by algebraic treatment. By representing a point on a parabola by an algebraic equation of the coordinates at the point, it was, by a historical metamorphosis, transformed from a geometrical object into an algebraic object: from being described by a rhetoric sequence (referring to a configuration), often along with a diagram (drawing), it now showed itself as an algebraic expression such as $y = x^2$, still though, along with natural language. By defining the geometrical properties of a curve, it was possible by the use of coordinates to find an algebraic representation of this curve[13], provided the algebraic manipulations could be coped with. As an example, from the defining property for a parabola as the loci for points with the same distance to a fixed point (*focus*) as to a fixed line (*directrix*)[14], an algebraic calculation provides the

[12] In Merriam-Webster's online dictionary (http://www.merriam-webster.com/dictionary), 'metamorphosis' refers to "a major change in the appearance or character of someone or something"

[13] In the embodied cognition literature this phenomenon is termed a *conceptual blend* (Lakoff & Nuñez, 2000, p. 48).

[14] In the *Conics* of Apollonius we find the corresponding properties of the hyperbola and the ellipse in propositions III.51 and III.52, respectively.

equation $x^2 = 4ay$ if the focus point is in (0, a) and the directrix is $y = -a$, using a current standard ON system. As the algebraic equation $y = x^2$ (or, with the use of set theory formalism, $\{(x,y) \in R^2 : y = x^2,\ x \in R\}$) does not in itself constitute an icon of the curve it generates by its interpretation into coordinates for points in a standard coordinate system, its status as a representation of a parabola is 'abstract', as expressed by Fried (2007, pp. 216-217):

> To understand our own mathematical approach to objects such as the parabola, then, it is essential to see how Descartes relocates mathematical thought from the system of ideas of Apollonius' world to a new system of ideas in which objects like the parabola are redefined and reinterpreted in terms of abstract relations (i.e. relations lifted up and away from special subject matter).

By this "relocation" of mathematical thought into a new "system of ideas", a historical metamorphosis of meaning took place. For example, when the parabola in a school curriculum is introduced algebraically as a quadratic function, the 'meaning' linked to the curve being more or less 'narrow' or 'wide' is "relocated" to the size of the coefficient of the quadratic term, sustained with a computational argument (for the same value of x, $3x^2 < 4x^2$) prior to its interpretation in a diagram. In the conic section definition, the increasing 'wideness' can be directly perceived as the cut with the plane is done further away from the vertex of the cone (a bigger parameter), where its circular base is wider.

The fact that Fermat reverted this process by finding the geometric counterparts starting with given algebraic equations (Eves, 1983, p. 265), also shows how the power of representations through semiotic invention and chaining may produce new and unexpected knowledge. Curves never before thought of, such as the 'parabolas of Fermat' (given by the equations $y^n = ax^m$), or in modern time fractals, could be 'described' by algebraic means and interpreted iconically when represented in the plane (or space) by coordinates. Descartes performs in the second book of his *Géomètrie* a classification of all possible curves produced by a quadratic equation in two variables, and opens the way for higher order equations. His analytic method is clearly expressed in the following lines:

> I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all points of those curves which we call ”geometric”, that is, those which admit of precise and exact measurement, must bear a definite relation to all points on a straight line, and that this relation must be expressed by means of a single equation. If this equation contains no term of higher degree than the rectangle of two unknown quantities, or the square of one, the curve belongs to the first and simplest class, which contains only the circle, the parabola, the hyperbola, and the ellipse; but when the equation contains one or more terms of the third or fourth degree in one or both of the unknown quantities (for it requires two unknown quantities to express the relation between two points) the curve belongs to the second class; and if the equation contains a term of the fifth or sixth degree in either or both of the unknown quantities the curve belongs to the third class, and so on indefinitely. (Descartes, 1954, p. 48)[15]

Curves corresponding to the equation $y^n = px$, $n > 2$, for example, are considered parabolas of higher order (Eves, 1983, p. 264). New meanings may also be created: when the properties of geometric objects are studied by work in the algebraic domain, the latter may serve as a means to better understand, or gain new knowledge, of the former (Bolea, Bosch, & Gascon, 1999). The symbolic language of algebra can thus be seen as a *didactic tool* to learn about the other domain, in this case geometry, fundamentally different (Bergsten, 2003).

As an example of Descartes' algebraic work, in performing the classification of the second degree equation, the following short passage gives a flavour of both his skill and enthusiasm for his new method[16]:

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[15] It can be observed here how Descartes still sticks to the tradition when interpreting second order terms like xy or xx geometrically (”rectangle”, ”square”), but calls higher order terms more freely as ”third or fourth degree”, and writes them with exponential notation.

[16] This enthusiasm also echoes in the very last words of the 1637 treatise: ”I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.” (Descartes, 1954, p. 240)

Again, for the sake of brevity, put $-\frac{2mn}{z}+\frac{bcfgl}{ez^3-cgz^2}$ equal to *o*, and

$\frac{n^2}{z^2}-\frac{bcfg}{ez^3-cgz^2}$ equal to $\frac{p}{m}$; for these quantities being given, we can represent

them in any way we please. Then we have $y=m-\frac{n}{z}x+\sqrt{m^2+ox+\frac{p}{m}x^2}$

(Descartes, 1954, p. 63)[17]

The quotes above serve as warrants for the use of the term *historical metamorphosis*, even though the paragraphs above are rather unhistorical in any other sense than by their presentation of an account of appearances of certain texts at different points in time in the history of mathematics. The focus is on the metamorphoses of meanings, residing beyond the representations given, in Peircean terms within the interpretant. In the case of *les anciens*, a term Descartes frequently uses as a reference for Greek mathematics during antiquity, a meaning of the sign parabola (*παραβολή*) could be inferred from a verbal description of a geometric construction. While the close affinity between the different conics has its origin in a gradual change within the construction (the angle of a cutting plane in relation to a cone, or the size of the lack or excess of an applied area) for *les anciens*, for Descartes it is a matter of the gradual change of size of coefficients in a quadratic algebraic equation in two variables.

I will only shortly touch upon some immediate as well as far reaching consequences of Descartes' *method*[18] over another 200 years, until another metamorphosis of the meaning of parabola relevant for education occurs. The historical metamorphosis of geometry into analytic geometry was paradigmatic. As soon as the sometimes heavy geometric analysis had been complemented with smooth algebraic calculus, the development of mathematics exploded. Not only the ancient

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[17] Descartes used the notation xx for x^2 and a special sign for equality.

[18] As well known, Descartes did not draw two perpendicular coordinate axes and used only positive coordinates. His *method* was made more available to the contemporary public by Van Schooten's 1649 Latin translation and commentary. A 'modern' coordinate system with perpendicular positive and negative x- and y-axes and an origin O one finds for example in Newton.

conics, now called quadratic or second degree curves, but also corresponding 3D second degree surfaces[19] could be systematically studied (as for example by Euler, who introduced coordinate transformations to find the canonical form representation; see Kline, 1972, p. 545-547), also in the general setting of quadric forms, which by matrix notation and eigenvalue theory were given a unified and systematic treatment during the 19th century (see e.g. Kline, 1972, pp. 799-812). The formal notation of analytic geometry was easy to expand into any 'dimension', as shown by the first printed publication focused on higher-dimensional point geometry by Cayley in 1843 and other early independent work by Grassmann and Schäfli (Eves, 1983, p. 415).

A computational metamorphosis

A cone can be physically realised in wood, and cut by a plane realised as a metal saw, leaving the cut surface open for observation. The shape of the contour, being produced by the cut, is being born and, with visual perception and reflection, takes on its own life as a cultural object (cf. Radford, 2007), a 2D curve, the parabola. Whether such 'birth' of the parabola happened according to the standard Menaechmus story or not (see footnote 6), in the writings of Apollonius, by such type of intersection of two 3D objects (a cone and a plane in space), a 2D object was constructed and named *παραβολή* (Greek for application, here of areas) along with an investigating of its geometric properties (some of which have been highlighted above). However, while thus being developed by means of technology (even if those 3D objects were only mentally conceived or drawn, i.e. by a 'mental technology'), the new product (the parabola) became and remained an intellectual object for over two millennia, in essence static as represented by drawings on paper or as algebraic equations[20]). Controlled and systematic animation required a computational technology that did not (yet) exist.

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[19] According to Kline (1972, p. 321), Fermat had already in 1643 anticipated coordinate descriptions of these 3D surfaces.

[20] The parabola also 'survived' as an object of study in schools thanks to, I would hypothesize, its practical usefulness stemming from some of its geometric properties.

With the advent of computer technology in the 20^{th} century, performing high speed numerical calculations paired with advanced screen technology, a user can not only plot a graph from a given algebraic formula but also study geometrical properties of the graph (as for example a parabola) as (seemingly connected) points on a screen by doing simple hand movements using the ‘click’ menu and ”drag mode” of a dynamic geometry (or other types of) software, to realise animation while keeping construction generated invariances intact. The geometric (iconic) object is again in focus, and this time directly available without the intermediate representation of algebraic formula, subject to manipulation like the algebraic equations implicitly used to describe it and put it on the screen. By a historical metamorphosis the parabola now has appeared as a *dynamic object* on a computer screen (one may call this a computational metamorphosis). Mathematics has given itself, with advanced hard- and software technology, one more didactical tool, possibly to better understand itself.

However, before appearing on the screen available for animation, the parabola studied need to be defined as a specific parabola, either by a formula or a geometric construction performable by the software tools. Its meaning has to be *anchored*. The semiotic means of objectification (Radford, 2003) thus heavily draw on the cultural history of mathematics by its transmitted material representations as well as (or including) technological artefacts. That the dynamical features of the ‘new’ representational tool (such as dynamic geometry software) can incur an impact on meaning can be seen in relation to the construction of the parabola presented in Figure 4. Using the drag mode, a user can *experience* how the point P is being ‘pressed down’ heavily towards the horizontal axis when the point E approaches A, and ‘pushed up’ strongly when E is moved far beyond B. Applying the equidistance construction of the parabola (see Figure 7), the perpendicular bisector used for the construction appears as a tangent to the parabola, and the reflection property can be immediately conceived by the drag mode, other invariants not mentioned. The construction shown in Figure 4, as well as the ‘standard’ one in Figure 7, are both easy to perform (provided the software has been instrumented) and can generate to learners self steered and controlled experiences on which meanings of the parabola ‘lurking’ beyond the representation used can be constructed.

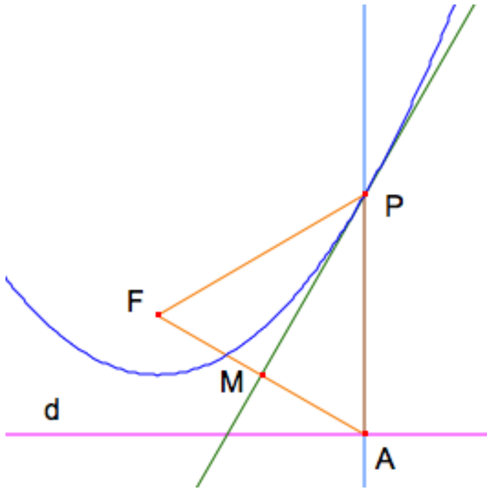

Figure 7. Equidistance construction of a point (P) on a parabola[21]

Discussion

The section produced by cutting a cone along a plane is, in Damerow's (2007) terms, a first-order representation of a real object. With some variation of how the cut is performed, an individual may develop a mental model of a curve called parabola. The establishment of properties, such as those outlined in the preliminaries above, involve second- (or higher-) order representations while employing deduced theorems from circle geometry. When some of these properties are used for constructing (defining) the parabola, another mental model of the curve is evoked, from which new higher-order representations can develop. With the historical transmission of these representations, a reconstruction and development of related meanings within the semiotic register of Euclidean geometry was possible. However, with the intermediate representation of distances as algebraic symbols (the *method* of Descartes), the superimposition of a coordinate system on the parabolic curve created a conceptual blend that constituted a paradigmatic metamorphosis of meaning. A higher-order representation of a geometric curve as an algebraic equation can refer to the original

[21] To the segment between the fixed point F (focus) and a variable point A on the fixed line d (the directrix), draw the perpendicular at the midpoint M, intersecting at point P the perpendicular to d at A. The point P is on a parabola by the equidistance definition. Note that the distance from F to d determines the shape of the parabola (though when scaled all parabolas are similar).

object only indirectly, via an intermediate higher-order representation, requiring a chain of connected mental models to have meaning.

The paradigmatic historical metamorphosis of the parabola pictured above may be conceived as an ‘algebraization’ of a mathematical organisation (or work; see the section on school mathematics below): “a mathematical work is algebraized if it can be considered as an algebraic model of another mathematical work, the system to be modelled” (Bolea et al., 1999, p. 142). Such kind of modelling should respond to all the techniques and technological questioning in the organisation being modelled as a whole (ibid.). This seems to be part of what Descartes aimed at with his analytic method (cf. the quote above from Descartes, 1954, p. 48) but it also took the source organisation further and his method proved self-generating for the development of mathematics.

By the ”parabolic” cases in 2 and 3 dimensions, i.e. $y = x^2$ and $z = x^2 + y^2$, the obvious (in some sense) ”parabolic” extension to n dimensions would be $y = x_1^2 + x_2^2 + x_3^2 + ... + x_{n-1}^2$, where $\bar{x} = (x_1, x_2, x_3, ..., x_{n-1}) \in R^{n-1}$. From a concrete action on a physical object – cutting a cone – the iconic ”section” of a parabola[22] by a historical metamorphosis returned in the shape of a symbolic representation in the semiotic register of algebra. By the development of this register, hitherto unknown mathematical ‘objects’ were hidden beyond transformation, variation and generalisation of this representational form. With the new eyes new things were found, and old things were looked at in new ways. As a consequence of this semiotic relocation, by the human quest for meaning, when images from sensual experience are not available (as in the case of multidimensional “geometry” in R^n, $n > 3$), the phenomenon of metaphorical mapping (Lakoff & Nunes, 2000, p. 43) comes into play, witnessed by the choice of names and diagrams most often produced for communication. Examples include the expression “unit ball” when referring to the set $\left\{\bar{x} \in R^n : x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 < 1\right\}$, or the use of the word “distance” between

[22] The name parabola, though, alludes to the idea of application of areas and not to cone cutting.

functions as elements in a Hilbert space (Bergsten, 1993)[23]. A similar claim was made in Bourbaki (1974) about the increased abstractness of mathematics, "to the extent that geometry has been transformed into an universal language for contemporary mathematics" (Bartolini Bussi, 2005, p. 40). Is this an indication of the impact of the "spirit of geometry" on meaning making in mathematics?

By the computational metamorphosis the iconic representation of a geometrical object turned dynamic. Although the individual can experience it as a direct control of the 'real' object, the object on the screen is recomputed by the software for each touch of the drag mode, thus produced by higher-order representations. That the object is only virtually real as a material higher-order representation does not prevent it from supporting mental models that carry meaning, where the dynamic feature of the virtual object allows the 'discovery' of invariances by the use of drag mode[24]. The virtuality of the object is irrelevant for the proof of the invariance by the discursive tools of Euclidean geometry.

Is it then meaningful to ask what a parabola *is*, beyond the representation given? Is there a "truth" hidden at the horizon, in line with the Peircean dream? What is it that holds the 'concept' of a parabola together, or what is, in Husserl's (1989) terms, the *ideal object* 'parabola'? What seems to remain invariant across the representations given is the iconic configuration (see Bartolini Bussi, 2005, p. 40, as quoted above). But the "spirit of geometry" cannot reside in this configuration per se but in the elaboration of the geometric properties 'embedded' in this configuration through its mode of construction – embedded relative the discursive part of a given mathematical organisation (such as Euclidean geometry, that René Thom may have had in mind). Where the 'real object' beyond the representation is, if anywhere, seems to have no relevance for the experience of meaning. What is taking place is a chain of re-descriptions by way of chains of higher-order representations and reconstructions of mental models.

[23] As a remark here, Apollonius' definition of a conic surface displays an image schematic character (the path schema; see Lakoff & Nunes, pp. 37-39, 141) in being thought-of generalised bodily based images.
[24] One example of such 'discovery' is given in the discussion of Figure 8 below.

The semiotic perspective shows how meaning is transferred through the representational forms to what the interpreter within a given cultural situation and background constructs from them. And the historical development of mathematics shows, as illustrated here by the example of the parabola, how the cultural meaning changes its face when new semiotic registers are developed and used on objects that used to be studied by other registers. Through a *semiotic chain* the intersection of a cone with a plane (i.e. an icon) has been linked to a quadric form (the notation of which is a symbol), represented by a symmetric matrix (symbol), and to electronic dots on a computer screen (together constituting an icon), constituting a move from first-order to second- and higher-order representations. However, new developments do not always replace old meanings with new but rather grow as new layers of signification outside the old. A process of a similar kind, at the individual level, has been described by Presmeg (2006) by a Peircean nested triadic model of semiotic chaining. The model, resonating with Radford (2006), may be applied also to a historical-cultural development of meaning, as the one of the parabola sketched above. Still, by the development of new representations, the process is not expected to end at some unified conception of what, in this case, a parabola is. In analysing the historical development of another mathematical object, the fundamental theorem of calculus, Jablonka and Klisinska (2011) conclude that one would expect not a convergence but even more diversity of meanings, and that it is in education rather than within the scientific discipline that efforts of unification are found.

An additional conception of Peirce may overarch, and in the realm of reasoning and problem solving integrate, the iconically based argumentation of Euclidean geometry as well as the symbolically based in analytic geometry, i.e. diagrammatic reasoning:

> all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts (Peirce, quoted in Dörfler, 2004, p. 7)

Also the manipulation of algebraic formulas is a work with icons, the patterns of formulæ: “These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra.” (ibid.; cf. the concept of *mathematical form* in Bergsten, 1999). This kind of diagrammatic reasoning is of such generality that it may function, for the individual, as a linking force between different mathematical domains, since it is not primarily focused on the referential function (or meaning) of the forms (representations).

The parabola in school mathematics

As well known, René Thom suggested that mathematics education should be founded on meaning rather than rigour. As if one could not have both. The German tradition of *Stoffdidaktik* within mathematics education seems to have aimed at precisely this with its emphasis on grounding teaching on *mathematical background theories*, ways of making *fundamental ideas* available to students at different age levels, and *Grundvorstellungen* in students (see e.g. Tietze, 1994; Bergsten, 2014). Principles used in the process of selection and analysis of the mathematical content to be taught include *elementarizing*, *exactifying*, *simplifying*, and *visualizing* (Tietze, 1994), as well as *complementarity* of theoretical and pragmatic aspects of knowledge (Otte & Steinbring, 1977). The present discussion tries to highlight the complementarity of meaning and rigour through an elaboration of historically transmitted forms of external representations as semiotic means of objectification, drawing on the analyses of Radford (2006) and Damerow (2007), as presented above. Reference has also been made to the *Anthropological Theory of the Didactic* (ATD; e.g. Chevallard, 1998), which will form the basis for the discussion of school mathematics in this section.

In this theoretical approach (ATD), two inseparable aspects of mathematical activity within an institution are identified, one practical (know-how) and one theoretical (know-why). The former consists of types of tasks/problems that are studied and techniques used to solve them, while the latter is formed by the corresponding discursive environment, i.e. issues of technology (the discourse/ ingredients related to the techniques) and theory (deeper justification), together constituting *mathematical organisations* or *praxeologies* (ibid.; see also Bosch &

Gascon, 2006). The ATD grew out from earlier elaborations on didactic transposition theory (e.g. Chevallard, 1991; see also Bergsten & Jablonka, 2008).

As an example, the task of finding the midpoint of a given circle may be solved by the technique of constructing midpoint perpendiculars to two chords in the circle, thus using technological ingredients such as chords to circles, justified by theorems in Euclidean geometry. Mathematical organisations can be considered at different levels, where a *punctual* mathematical organisation of a specific type of problem and technique used to solve it can be embedded in a *local* mathematical organisation with the technology available for using those techniques. Some local mathematical organisations may build on the same theoretical discourse to form a *regional* mathematical organisation (Bosch & Gascon, 2006). By the process of didactical transposition, different levels of co-determination (i.e. the level of society, school, pedagogy, discipline, area, sector, theme, and subject) set didactic constraints on classroom activity. The classroom activity of a teacher is more or less restricted to the last two levels, with the mathematical scope of the three preceding levels. (ibid.)

Given this theoretical framing, students' first acquaintance and work with the parabola is constrained by the local mathematical organisations previously met and their integration or fragmentation, and in which mathematical organisation it is embedded. Levels of justification also vary with the didactic transposition induced by the curriculum and the kinds of problems studied. The meanings made 'available' to students can thus vary accordingly.

Examples of didactic transpositions of the parabola

For the upper secondary science student in Sweden in the 1960s, a parabola was the geometric locus of points equidistant to a given point (*focus*) and a straight line (*directrix*), a property which was immediately elaborated by the algebraic register of analytic geometry to the formula $y^2 = 4ax$, upon which a systematic treatment of a number of geometric and algebraic properties was based (e.g. Sjöstedt & Thörnqvist, 1963, pp. 62-76)[25]. The study of the parabola was embedded in a local mathematical

[25] The approach of Apollonius was also included in this textbook, for optional deepening studies.

organisation of analytic geometry, including techniques of Euclidean and coordinate geometry, as well as elementary algebraic equation solving. Techniques from the study of functions, such as the derivate, were not used.

Later, in the 1980s and 1990s, a metamorphosis of meaning had been introduced: for the ”same” student the parabola was now introduced as an algebraically defined curve of second degree, $y = x^2$, the shape of which was plotted, based on a table of values (e.g. Björk, Borg, Brolin & Ljungström, 1990, pp. 264-274). There was only little or no discussion about any geometric properties of the curve (such as equidistance or reflection), other than the symmetry (based on the algebraic argument that $(-x)^2 = x^2$) and the vertex point. In ‘applied’ textbook tasks, specific algebraic equations for the curves involved were mostly provided. Tangents and optimization problems were then treated by means of the derivative. The study of the parabola was embedded in a local mathematical organisation of functions. Techniques from Euclidean geometry were used only casually, as well as analytic geometry apart from the basic idea of coordinate representation as a means for plotting the graph of a function. The textbook by Bergendahl, Håstad and Råde (1975) marks a transition state where the parabola is introduced as a second degree polynomial curve (embedded in a mathematical organisation of functions) in the main text but accompanied by an appendix (optional) chapter about the conic sections framed within analytical geometry.

For university students, shortly, the acquaintance with quadratic surfaces (such as the paraboloid) and the corresponding plane curves, is accomplished by the study of quadratic forms, embedded in the domain of linear algebra. However, the application of integrals on rotational volumes for the same mathematical objects is often done in isolation from linear algebra within the local mathematical organisation of calculus (based on a discursive technology involving real numbers, functions and limits of functions).

Objectification of knowledge

In many educational situations, the student is presented objects of learning previously not known (or encountered), as might be the case with the parabola in a mathematics class. For the student to become aware of this object the teacher might use "semiotic means of objectification – e.g. objects, artifacts, linguistic devices and signs that are intentionally used by the individuals in social processes of meaning production" (Radford, 2002, p. 14). The drawing of a parabola may seem the optimal such means but is by itself void of meaning besides its appearance. It is necessary to go beyond the representation given to find meaning. The same remark holds, even more so, for an algebraic representation, such as $y = x^2$ or $y^2 = 4ax$, of the parabola. In the case of the icon, a defining property such as equidistance (focus-directrix) does not only give the shape of the icon (the curve) but also provides a basis for further explorations and analyses of the object as a support for attaching 'meaning' to it. Using another defining non-algebraic property, such as the application of areas as outlined in in Figure 4, generates the same shape but offers another basis for further explorations, and 'meanings':

> the figural representations of conics […] are invariant in time and hence not subject to historical changes. What are changed are the way of generating conics, the way of looking at them, and the way of studying them (Bartolini Bussi, 2005, p. 40).

The objectification process for the learner diverges between the options, even more so by providing an algebraic formula as the main means of objectification, leading into a different mathematical organisation by the different techniques available. In a learning setting, such options to relate to other representations or other semiotic registers are restricted by the overall didactic situation, as a consequence of the didactical transposition that has taken place. By those constraints, students' work may remain at the punctual level, resulting in concept images often too vague for successful problem solving in situations spanning over more than the restricted punctual or local mathematical organisation that has been covered in class.

An example of pedagogy

With reference to work by Vygotsky and Bachtin, among others, Boero et al. (1997) have designed an "educational situation" they call the "voices and echoes game" that seems to be akin to the theoretical outlines by Radford (2006) and Damerow (2007) above on the relationship between the individual and the socio-historical knowledge development, with a specific focus on the role of representations in the meaning making processes. By encountering historical approaches in texts that represent "historical leaps" in the development of mathematics, students can produce *echoes* to these *voices* from the past through work on specific tasks. These echoes can be individual or collective depending on the organisation of the classroom activities. The aim of the activity is to find ways to organise the mediation and appropriation of theoretical knowledge.

Historical metamorphoses of meanings as discussed in this chapter may be useful as bases for the design of such 'specific tasks', to provide semiotic means of objectification of mathematical objects. What may be called *bridging tasks*[26] link well to the present discussion, provided that the techniques and technology of the different mathematical organisations are available to the students. Such task would create a *bridge* between (historically based) different representations of a mathematical object (a conversion between different semiotic registers or a treatment within the same semiotic register), a bridge that would have the potential to *carry meanings* across representations and mathematical organisations. An obvious example related to the parabola is to show that the curve defined algebraically by the equation $y = x^2$ (where x and y are Cartesian coordinates for points on the curve) has the equidistance property, and vice versa. An equally obvious follow up from this task is the demonstration of the reflection property of a parabola in the different mathematical organisations related to these two definitions. In the case of the algebraic definition, the technology of the mathematical organisation for the study of functions may include the derivative. Here, meanings available in the geometric case may provide a background gaze to the technicalities of the algebraic work needed to solve the task within the study of functions algebraically defined, where the landscape of meaning would remain barren without feeding it with the "spirit of geometry".

In a bridging task involving a comparison of the two geometric constructions of the parabola shown in Figure 4 and Figure 7, respectively, the representations are both

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[26] The label "bridging task" is currently used in different educational contexts, as for example literature studies where the aim is to link a text to its historical/ social/ cultural context.

within the same semiotic register. One question to ask in such comparison is whether the parabola produced by the application of areas in Figure 4 has the equidistance property on which the construction of the curve in Figure 7 is based (to justify the use of the name *parabola* also for this curve). Leaving this "pleasure of discovery" to the reader, a similar but more comprehensive bridging task will instead be discussed here in more detail:

> ***Example***. *How can one find the axis and the focus point of a parabola constructed by the application of areas as in Figure 5a?*

By this example a bridge can be established between the parabola as defined by the equidistance property, where the focus point is given, and as defined by the application of areas in the non-rectangular case in Figure 5a (that is within the same mathematical register and organisation of Euclidean geometry). The example raises some critical issues on meaning and its dependence on a reference knowledge base.

Figure 5a does not itself offer much advice on how to proceed. However, Figure 5b shows that the line through A and D passes the midpoints of parallel chords to the parabola, and thus is a diameter to the parabola, parallel to the axis (drawing on Apollonius). This means that a chord PP' perpendicular to this diameter can be drawn which will also be perpendicular to the axis. Consequently, the perpendicular bisector to PP' must produce the axis, meeting the parabola at its vertex V (see Figure 8 where MV is the axis). To proceed with the problem it is thus necessary to bring in additional theoretical knowledge, such as definitions and properties of diameters and the axis of a parabola, and knowledge about where on the axis the focus point is located.

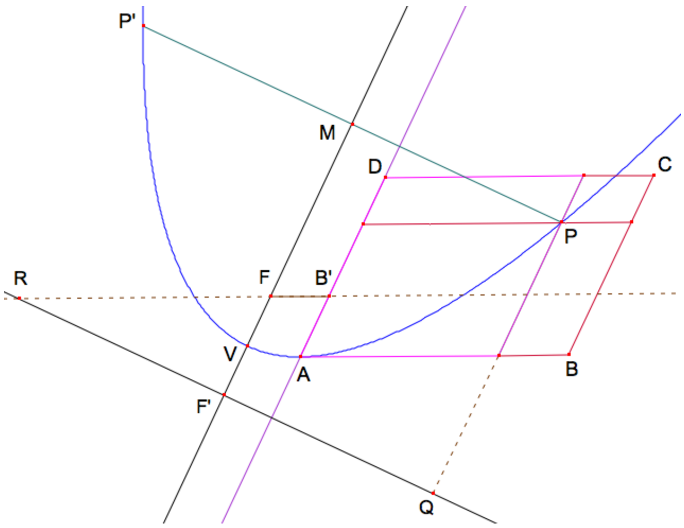

Figure 8. Construction of the axis and focus of a parabola as defined in Figure 5a

As shown in the preliminaries above, the distance from the vertex to the focus point is one fourth of the parameter (equal to the length of the *latus rectum*). Select, then, the point B' on the line through A and D so that AB' is a fourth of AB. Using the drag mode of the software to 'straighten up' the parallelogram ABCD to a rectangle shows that the line through B' parallel to AB meets the axis at the point F, the focus point of the parabola. Select F' on the extended axis so that VF' is equal to VF. The line perpendicular to the axis at F' is the directrix to the parabola (with PF equal to PQ; the tangent at P to the parabola is the perpendicular bisector to F and Q). It can also be observed (using the drag mode) that the line through A and V meets the directrix and the line through B' and F at their point of intersection R. Here, this property does not seem to add more meaning to the construction, though in geometry invariance is always compelling. Using the drag mode (e.g. varying the angle of the parallelogram ABCD) other invariances can be 'discovered'. For example, the circle with centre at A and passing through B' also passes through F and touches RQ. This means that both F and the directrix can be constructed simply by drawing this circle. Alternatively, the circle with centre at B' having a radius of twice the length of AB' (and thus a diameter equal to the parameter AB) meets the parabola at two points defining a chord parallel to AB and passing through F.

The constructions above for finding F thus draw heavily on moving between theoretical and figural (or diagrammatic, in the sense of Peirce) reasoning, thus constituting *an experience of meaning production* (Laborde, 2005). What seems to work figuratively (confirmed by the use of drag mode), however, is not always theoretically evident, implying a need of a theoretical search for meaning. In this case, the claim that F so constructed is, indeed, the focus point, is such an issue. This could lead to the search for meaning beyond other representations. One option to find F draws on knowledge about the *latus rectum*: Construct the point S on the line through P and P' so that the length of MS is twice the length of MV (see Figure 9). The line through V and S then meets the parabola at the endpoint T of latus rectum (by similarity), and the perpendicular to the axis through T will thus meet the axis at the focus point F (producing the right half FT of latus rectum in Figure 9). This construction is potentially rich in meaning but requires that the software can produce intersection points with the parabola.

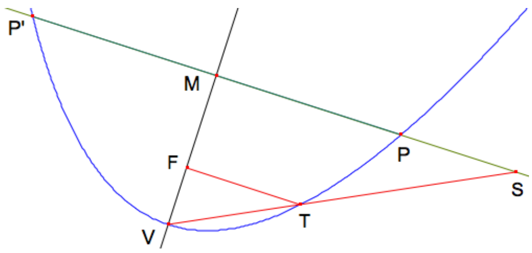

Figure 9. Construction of the focus point of a parabola with a given axis

Bridging tasks may also be designed to draw more explicitly on the history of mathematics, as in the teaching experiment described in Bartolini Bussi (2005) where historical mathematical machines are used for the construction of the parabola along with the pre-Apollonian definition of conics as sections of a right-angled cone, or as in examples of the "voices and echoes game" described in Boero et al. (1997; 1998).

An individual involved in mathematical work today has potential access to all historically transmitted material representations of a mathematical object. This holds, of course, for teachers and students at all levels of education, as well as educational policy makers. There is therefore always a choice when a curriculum is to be designed regarding which mathematical organisation is seen as most appropriate in relation to the specific educational goals and socio-cultural conditions. It is a question of curriculum development to design school mathematics education to support students' development of conceptual models and problem solving techniques so that their space of reference is not constrained to the punctual level. By linking the different representations of the objects of learning (e.g. the parabola) to more general mathematical organisations into which they can be embedded, the representational flexibility needed for solving more non-routine problems can be developed through the refractions of the complementarity between meaning and rigour this evokes.

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