CHAPTER 1.5 – WORKSHEET: ALGEBRAIC IDENTITIES

1. Theory Summary & Definitions

Identity: An identity is defined as any equality that contains variables and holds true for all values of its variables. For example, the equality a(β + γ) = aβ + aγ is an identity.

Remarkable Algebraic Identities

1. Square of a Sum: $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$

The right-hand side of the equality is called the expansion of the square.

Proof: $(\alpha+\beta)^2=(\alpha+\beta)(\alpha+\beta)=\alpha^2+\alpha\beta+\alpha\beta+\beta^2=\alpha^2+2\alpha\beta+\beta^2$

2. Square of a Difference: $(\alpha-\beta)^2=\alpha^2-2\alpha\beta+\beta^2$

Proof: $(\alpha-\beta)^2=(\alpha-\beta)(\alpha-\beta)=\alpha^2-\alpha\beta-\alpha\beta+\beta^2=\alpha^2-2\alpha\beta+\beta^2$

3. Cube of a Sum & Cube of a Difference:

a) $(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

b) $(\alpha-\beta)^3=\alpha^3-3\alpha^2\beta+3\alpha\beta^2-\beta^3$

Proofs:

$(\alpha+\beta)^3=(\alpha+\beta)^2(\alpha+\beta)$ =

a) $(\alpha^2+2\alpha\beta+\beta^2)(\alpha+\beta)=\alpha^3+\alpha^2\beta+2\alpha^2\beta+2\alpha\beta^2+\alpha\beta^2+\beta^3$ =

$\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

$(\alpha-\beta)^3=(\alpha-\beta)^2(\alpha-\beta)$ =

b) $(\alpha^2-2\alpha\beta+\beta^2)(\alpha-\beta)=\alpha^3-\alpha^2\beta-2\alpha^2\beta+2\alpha\beta^2+\alpha\beta^2-\beta^3$ =

$\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

4. Difference of Squares (Product of Sum by Difference): $(\alpha+\beta)(\alpha-\beta)=\alpha^2-\beta^2$

Proof: $(\alpha+\beta)(\alpha-\beta)=\alpha^2-\alpha\beta+\alpha\beta-\beta^2=\alpha^2-\beta^2$

Important Remarks & Notes

a) An identity is an equality that holds true for every value of its variables. It is NOT an equation that happens to have infinitely many solutions.

b) The relation $(\alpha+\beta)^2=\alpha^2+\beta^2$ does NOT hold in general. It holds ONLY when at least one of a or b equals to 0.That is $(\alpha+\beta)^2=\alpha^2+\beta^2$ if and only if $\alpha=0$ or/and $\beta=0$

c) If $\alpha^2+\beta^2=0$ for real numbers a, b, then a = 0 and b = 0.

d) It holds that $(\alpha-\beta)^2=(\beta-\alpha)^2$, whereas $(\alpha-\beta)^3\neq(\beta-\alpha)^3$.

Exercise Methodology

1. To prove an equality (identity), we start from the left-hand side (first member), perform algebraic operations, and arrive at the right-hand side (second member).

2. Alternatively, we perform algebraic operations on both members independently until we reach the exact same expression.

3. To show that an algebraic expression is independent of a variable x, we perform all indicated operations until we arrive at a result that does not contain x.

1. Characterize the following statements as True (T) or False (F):

1. $\alpha^2+\beta^2=0$
2. $(x-y)^2=(y-x)^2$
3. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
4. If $\alpha^2+\beta^2=0$, then α = 0 or β = 0.
5. If $\alpha^2+\beta^2=2\alpha\beta$, then α = β.
6. If $\alpha^2+\beta^2-(\alpha+\beta)^2=0$, then α = 0 or β = 0.
7. It holds : $-(\alpha+\beta)(\alpha-\beta)=\beta^2-\alpha^2$
8. It holds : $(x-y)^2=x^2-2x(-y)+(-y)^2$

A2. Select the correct answer in the following questions:

1. If α - β = 6, then the value of the expression $K=\alpha^2+\beta^2-2\alpha\beta+3$ is equal to:
   a. 0 b. 39 c. 33 d. 81
2. If x + y = 20 and $x^2 - y^2 = 80$, then the value of the expression Λ = y - x is equal to:
   a. 4 b. -4 c. 1 d. 10
3. If $x-\frac{1}{x}=3$, then the expression $x^2+\frac{1}{x^2}$ is equal to:
   a. 9 b. 11 c. -11 d. -9

B1. Find the expansions of the following products:

1. $(4x-y)(4x+y)$
2. $(a+2\beta)^3$
3. $(2x+3x^2)^2$
4. $(\frac{2}{3}x^2-4x)^2$
5. $(\frac{3}{4}a^2-4a)^3$
6. $(3x^2-y)(3x^2+y)$
7. $(x-y)(x+y)(x^2+y^2)$

B2. Perform the operations:

α) $(3\alpha-\beta)^2-3(4\alpha+5)(4\alpha-5)$

β) $(3\alpha-3\beta)^2-(3\alpha+2\beta)^2+3\alpha(\beta-1)$

γ) $(1-x)(1+x)+(x-1)^2+2(x-1)$

B3. Given the polynomial $P(x)=3x^2-5x+1$. Find:

α. P(x – 10), **β.** P(3x – 2) **γ.** P(-x + 1) - 2x

B4. Prove that the following polynomials are independent of x:

α) $P(x)=(x^3-1)^2+(x^3+1)^2-2(x^3-1)(x^3+1)$

β) $Q(x)=(x^2+1)^3+3(x^2+1)^2(1-x^2)+3(x^2+1)(1-x^2)^2+(1-x^2)^3$

B5. Prove the following identities / equalities:

α) $(\alpha+\beta)^2-(\alpha-\beta)^2=4\alpha\beta$

β) $(2026+\frac{1}{2026})^2-(2026-\frac{1}{2026})^2=4$

γ) $(3\kappa+2)^2-(2\kappa+3)^2=5(\kappa-1)(\kappa+1)$